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Pulsar Magnetospheres

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Pulsar magnetospheres

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The energy loss from the neutron star – as inferred from the secular increase in rotation period – is much greater than that emitted in either the radio or the other observed wavelengths. A primary motivation of magnetospheric theory is to trace the mode in which this energy and the associated angular momentum are in fact carried off from active pulsars. This review concentrates on the special case in which the magnetic and rotation axes are aligned. Electrons emitted from the polar caps are accelerated to highly relativistic energies by the electric force and simultaneously pick up angular momentum from the magnetic torque. Some process of angular momentum dissipation occurring beyond the light-cylinder is then required, both to yield the continuous spin-down of the star, and also to allow the electrons to cross magnetic field lines and so complete their circuits back to the star. Within the framework of classical physics, this could occur if most of the spin-down energy is lost through incoherent photon emission in an equatorial domain beyond the light-cylinder, but this would generate γ-radiation far in excess of that observed. Transport away of the angular momentum via a relativistic wind requires the generation of a quasi-neutral plasma. Gamma-rays emitted by outflowing electrons will produce electron–positron pairs in the strong magnetic field near the star, and highly energetic electrons returning to the star may also generate a mixed plasma by pair production or by surface spallation. Coupling with the circulating primary electron current may then ensure that the dominant angular momentum loss is via the wind rather than through photon emission.

1. Introduction

The first model for the pulsar magnetosphere predated the actual discovery of pulsars: in a prophetic paper, Pacini (1967) adapted the solution constructed earlier by A. J. Deutsch for the electromagnetic field of an obliquely rotating star *in vacuo* with its magnetic dipole moment $\boldsymbol{p}$ inclined to the rotation axis $\boldsymbol{k}$ by the angle χ. Far from the star the dominant terms are those forming the wave field emitted by the perpendicular dipolar component $p \sin\chi$, rotating with the angular frequency $\alpha = 2\pi/P_1$ with P_1 the period in seconds. The energy transport per second is

$$(2p^2\alpha^4/3c^3)\sin^2\chi = (B_s^2 R^6 \alpha^4/6c^3)\sin^2\chi, \tag{1}$$

where R is the stellar radius and B_s the polar field strength. If χ is assumed not very small, then the equating of (1) to the inferred energy loss $-I\alpha\dot{\alpha} = 4\pi^2 I\dot{P}_1/P_1^3$ from a star of moment of inertia I with the observed period increase $\dot{P}_1$ yields the canonical value of $B_s \approx 10^{12}$ G (Gold 1969). A corollary is that the energy loss vanishes as $\chi \to 0$: the aligned model *in vacuo* is dead.

The gravitational field of the neutron star ensures that for any reasonable temperature, the scale-height is so small that a thermally supported atmosphere would exponentiate down to virtually zero density within a few centimetres, so apparently justifying the assumption of not just a dynamical but also an electromagnetic vacuum. The axisymmetric model would then be like a dynamo on open circuit, with the rotation of the highly conducting neutron star crust generating enormous potential differences (of the order of $10^{17}/P_1$ V!) over the surface, but with no opportunity for currents to flow. The essence of the Goldreich–Julian ('GJ') critique (1969) is that the system will in fact build up its own conducting leads. The axisymmetric vacuum model has an external quadrupole electric field $\boldsymbol{E}$ with a component $\boldsymbol{E}_{\|}$ along $\boldsymbol{B}$ of the same order at the star's surface as the internal field $\boldsymbol{E} = -(\alpha\boldsymbol{k}\times\boldsymbol{r})\times\boldsymbol{B}/c$. The consequent normal discontinuity in $\boldsymbol{E}$ then requires a surface charge density, which is subject to very large outward-acting unbalanced electrical forces. Thus a charged magnetosphere is spontaneously built up, and if there is continuous current flow into and out of the star, the aligned model is no longer dead but suffers a steady loss of rotational energy and associated angular momentum. The argument applies also to the oblique case, with the Deutsch–Pacini energy loss supplemented by the work done on the magnetosphere charges by the electric field. The challenge to the theorist is to describe in detail the mutually interacting electromagnetic and particle fields. This report is devoted primarily to the steady, axisymmetric problem.

2. An illustrative classical model

The poloidal magnetic field is described by the flux function $P(\tilde{\omega}, z)$:

$$\boldsymbol{B}_{\mathrm{p}} = -\nabla\times(P\boldsymbol{t}/\tilde{\omega}) = -\nabla P\times\boldsymbol{t}/\tilde{\omega}, \tag{2}$$

where $\boldsymbol{t}$ is the unit azimuthal (toroidal) vector and $(\tilde{\omega}, \phi, z)$ are cylindrical polar coordinates. In a steady state the curl-free electric field is conveniently written as

$$\boldsymbol{E} = -\alpha\tilde{\omega}\boldsymbol{t}\times\boldsymbol{B}_{\mathrm{p}}/c - \nabla\psi \tag{3}$$

so that the electric potential

$$\phi = -\alpha P/c + \psi. \tag{4}$$

Within the rigidly rotating, perfectly conducting stellar crust the 'non-corotational' part ψ of the potential is a constant. The simplest alternative to the vacuum hypothesis for the surroundings is the GJ hypothesis, by which charges originating in the star distribute themselves so as to short out the component $\boldsymbol{E}_{\|}$, whence by (3), $\boldsymbol{B}\cdot\nabla\psi = 0$. Provided there are no vacuum gaps between the star and the point considered (cf. §4 below), the constant value of ψ within the star is then propagated into the magnetosphere: the vacuum condition of zero charge-current density is replaced by the 'corotational' electric field $\boldsymbol{E} = -\alpha\tilde{\omega}\boldsymbol{t}\times\boldsymbol{B}/c$, maintained by the charge density

$$\rho_{\mathrm{e}} = \nabla\cdot\boldsymbol{E}/4\pi = -(\alpha/2\pi c)\,[B_z - \tfrac{1}{2}\tilde{\omega}(\nabla\times\boldsymbol{B})_{\phi}]. \tag{5}$$

Provided $E < B$, a charge in the orthogonal $\boldsymbol{E}$ and $\boldsymbol{B}$ fields is subject to the drift $c(\boldsymbol{E}\times\boldsymbol{B})/B^2$. If there is no poloidal current $\boldsymbol{j}_{\mathrm{p}}$, the magnetic field remains purely poloidal, and this drift reduces to the corotation velocity $\alpha\tilde{\omega}\boldsymbol{t}$ for all charges within the light-cylinder (LC). Corotation beyond the LC being forbidden by special relativity, one can again define an electrodynamically dead model, with the corotating GJ charge density filling the whole domain with the LC, and a vacuum beyond. The Ampère equation yields

$$(\nabla\times\boldsymbol{B})_{\phi} = 4\pi\rho_{\mathrm{e}}\,\alpha\tilde{\omega}/c, \tag{6}$$

so that (5) and (6) become

$$\rho_{\mathrm{e}}\left[1-\left(\frac{\alpha\tilde{\omega}}{c}\right)^2\right]=\frac{-\alpha B_z}{2\pi c}=\frac{\alpha}{2\pi c}\frac{\partial P}{\tilde{\omega}\partial\tilde{\omega}} \tag{7}$$

and

$$\nabla^2 P[1-(\alpha\tilde{\omega}/c)^2]=2\partial P/\tilde{\omega}\partial\tilde{\omega}. \tag{8}$$

The solution of (8) subject to a dipolar singularity at the origin and with $B_z=0$ at the LC illustrates how the magnetospheric currents distort the field significantly from the vacuum dipole as the LC is approached (Michel 1991; Mestel & Pryce 1992). However, the discontinuities in $E_{\tilde{\omega}}$ and B_z across the LC imply a surface charge-current distribution with corresponding unbalanced Maxwell stresses: this dead model is again dynamically unacceptable (Mestel & Wang 1982).

Axisymmetric dead models have been constructed numerically by Krause-Polstorff & Michel (1985), satisfying the condition $\boldsymbol{E}\cdot\boldsymbol{B}=0$ within regions containing charge, but with large vacuum gaps. There are corotating negatively charged regions above the polar caps, and a super-rotating positively charged region around the equatorial region. The positive region is linked with the star by field lines passing through a gap along which $\boldsymbol{E}\cdot\boldsymbol{B}\neq 0$, so allowing deviation from corotation with the star. The models all have a large positive charge, and it is questionable whether they would in practice survive conversion into live models following either discharging by the interstellar medium or electron–positron pair production under the strong $\boldsymbol{B}$-aligned electric fields in the gaps.

Live models are likely to contain dead zones (analogous to those in the coronae of ordinary stars), contiguous with zones of poloidal current flow. We concentrate on the aligned rather than the counter-aligned case, so that the electric field is of the sign that will draw out electrons rather than ions. For the moment the problem is treated within the framework of classical physics, without pair production, so that the steadily outflowing electrons form a convection current, with the poloidal current $\boldsymbol{j}_{\mathrm{p}}$ defined by the stream function S:

$$\boldsymbol{j}_{\mathrm{p}}=\rho_{\mathrm{e}}\boldsymbol{v}_{\mathrm{p}}=-\nabla S\times\boldsymbol{t}/\tilde{\omega} \tag{9}$$

with ρ_{e} given by $\nabla\cdot\boldsymbol{E}/4\pi$. Rather than making the GJ approximations $\boldsymbol{E}\cdot\boldsymbol{B}=0$, it is preferable to retain inertial terms in the dynamical equations (though gravity is justifiably ignored); the validity of the GJ approximation in any domain can then be assessed. Thus for a cold, dissipation-free gas, the energy integral is

$$-e\phi+\gamma mc^2=-e\phi^*(S) \tag{10}$$

and the angular momentum integral

$$eP/c+\gamma m\Omega\tilde{\omega}^2=eP^*(S)/c, \tag{11}$$

where Ω is the local angular velocity. Equation (11) shows that the electrons do not move strictly along the field, but suffer an inertial drift: to increase its angular momentum $\gamma m\Omega\tilde{\omega}^2$, an electron must have a component of poloidal velocity normal to $\boldsymbol{B}_{\mathrm{p}}$ in order that the magnetic field exert the required torque. Together with the definition (4) of ψ, equations (10) and (11) combine into

$$\Gamma\equiv\{\gamma[1-(\Omega\tilde{\omega}/c)(\alpha\tilde{\omega}/c)]-e\psi/mc^2\}=\Gamma(S). \tag{12}$$

The 'relativistic slingshot' term $\gamma(\Omega\tilde{\omega}/c)(\alpha\tilde{\omega}/c)$ is a consequence of the inertial drift, which takes the electrons to points with a higher corotational potential $-\alpha P/c$. The actual deviation from strict flow along $\boldsymbol{B}$ is by (11) of order $\gamma\Omega/(eB/mc)$, which remains very small until the gas becomes highly relativistic (see below).

The current (9) generates by Ampère's law a toroidal field component $\boldsymbol{B}_{\mathrm{t}} \equiv B_{\phi}\boldsymbol{t}$ given by

$$B_{\phi} = -4\pi S/c\tilde{\omega}, \tag{13}$$

which in turn exerts a poloidal Lorentz force component $-e\boldsymbol{v}_{\mathrm{p}}\times\boldsymbol{B}_{\mathrm{t}}/c$. As in standard stellar wind theory, the steady-state angular velocity normally adjusts itself so as to lag behind the stellar rotation α. One can write both poloidal and toroidal velocities in this quasi-MHD flow in a form that takes account of inertial drifts:

$$\boldsymbol{v}_{\mathrm{p}} = \kappa\boldsymbol{B}_{\mathrm{p}}^{*} = \kappa(-\nabla P^{*}\times\boldsymbol{t}/\tilde{\omega}) \equiv \kappa[\boldsymbol{B}_{\mathrm{p}} - (mc/e)\nabla\times(\gamma\boldsymbol{v}_{\mathrm{t}})] \tag{14}$$

with the generalized flux function

$$P^{*} \equiv P + \gamma mc\Omega\tilde{\omega}^{2}/e = P^{*}(S), \tag{15}$$

and $$\Omega\tilde{\omega}\boldsymbol{t} = \boldsymbol{v}_{\mathrm{t}} = \kappa\boldsymbol{B}_{\mathrm{t}}^{*} + \alpha(S)\,\tilde{\omega}\boldsymbol{t} = \kappa[\boldsymbol{B}_{\mathrm{t}} - (mc/e)\nabla\times(\gamma\boldsymbol{v}_{\mathrm{p}})] + \alpha(S)\,\tilde{\omega}\boldsymbol{t}, \tag{16}$$

where by (9) and (14) the scalar κ is related to P^{*} or S by

$$\rho_{\mathrm{e}}\kappa = \mathrm{d}S/\mathrm{d}P^{*}. \tag{17}$$

As long as these equations hold all the way along the trajectory from the rigidly rotating star to the point considered, $\alpha(S)$ coincides with the stellar rotation α.

Because of the lag of Ω behind α, there exists a subrelativistic flow, closely following each poloidal field line, which can penetrate the LC. The condition that γ remains finite at the LC fixes the velocity with which the electrons leave the star, typically $\approx \frac{1}{2}c$. The precise behaviour of the flow beyond the LC depends on the detailed structure of the magnetic field lines, which strictly can be determined only simultaneously with the density-velocity fields of the particles. However, with virtually any plausible guess for the field structure, one finds that over much of the field the quasi-MHD flow breaks down somewhere beyond the LC, with γ becoming formally infinite when the electric field $\boldsymbol{E} = -\alpha\tilde{\omega}\boldsymbol{t}\times\boldsymbol{B}_{\mathrm{p}}/c$ becomes equal in magnitude to the total magnetic field $(B_{\mathrm{p}}^{2}+B_{\mathrm{t}}^{2})^{\frac{1}{2}}$. When γ approaches the value $\gamma_{\mathrm{c}} = (eB/mc)/\alpha$ (the ratio of the local non-relativistic gyro-frequency ω_{g} to the stellar rotation), the inertial drift terms are no longer small, and simultaneously the non-corotational potential ψ, given by equation (12), becomes comparable with the corotational part $-\alpha P/c$. Near the LC,

$$\gamma_{\mathrm{c}} \approx 2.6\times10^{7}(B_{\mathrm{s}}/10^{12})\,(R/10^{6})^{3}/P_{1}^{2},$$

where the rough dipolar approximation has been used for the field strength.

A deviation from flow along $\boldsymbol{B}_{\mathrm{p}}$ lines is to be welcomed, for clearly a steady state requires that the outflowing electron current be balanced by a return current. But the non-dissipative flow of charges of one sign is subject to the continuity condition (17) which keeps $\rho_{\mathrm{e}}\kappa$ constant all the way along the streamlines P^{*} (or S) = const., however much or however little they deviate from the flux lines P = const. Since ρ_{e} retains its sign, so does κ. Outflowing subrelativistic electrons from a polar cap are described by equation (14) with the inertial term negligible and $\kappa > 0$. Because of the equatorial symmetry of the problem, the returning current has to flow in along a field line in the same polar cap. With κ forced to remain positive, in the return flow the inertial term in (14) must dominate over $\boldsymbol{B}_{\mathrm{p}}$, requiring absurdly large γ-values. In any case, the purpose of the model is to yield a pulsar that spins down, and which

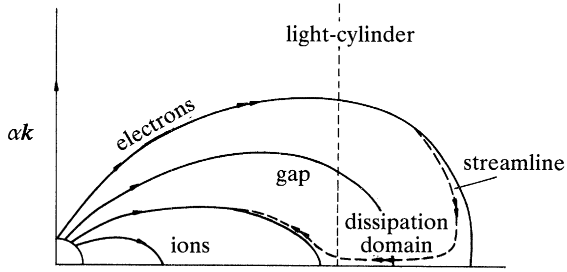

Figure 1. Schematic diagram of the classical model.

therefore must be subject to a dissipative process that carries off energy and angular momentum. The constraint $\rho_e \kappa$ = const. will then be automatically broken, and the difficulty just noted will vanish.

In fact, even in the slower pulsars with $P_1 \approx 1$ s, particles which have acquired γ-values close to γ_c – so that inertial drifts are significant – will also be emitting powerful incoherent γ-radiation, strongly beamed in the direction of motion. If $\mathbb{P}$ is the radiated power per particle, the associated linear momentum loss is $-(\mathbb{P}/c^2)\,\boldsymbol{v}$, as is seen by making a Lorentz transformation from the particle rest frame to the inertial frame. The energy and angular momentum integrals (10) and (11) are now replaced by

$$\boldsymbol{v}\cdot\nabla(\gamma mc^2 - e\phi) = -\mathbb{P} \tag{18}$$

and

$$\boldsymbol{v}\cdot\nabla(\gamma m\Omega\tilde{\omega}^2 + eP/c) = -(\Omega\tilde{\omega}^2/c^2)\mathbb{P}, \tag{19}$$

and (12) becomes

$$\boldsymbol{v}\cdot\nabla\left[\gamma\left[1-\frac{\Omega\tilde{\omega}}{c}\frac{\alpha\tilde{\omega}}{c}\right]-\frac{e\psi}{mc^2}\right] = \frac{-\mathbb{P}}{mc^2}\left[1-\frac{\Omega\tilde{\omega}}{c}\frac{\alpha\tilde{\omega}}{c}\right]. \tag{20}$$

The first term on the right of equation (20) represents the direct loss of energy by radiation. The associated angular momentum loss in equation (19) implies a 'dissipative' drift to field lines of higher corotational potential $(-\alpha P/c)$, and so to a gain in energy, analogous to the inertial drift that yields the slingshot term in (12) and (20). Going back to first principles: the energy source is the potential variation $-\alpha P/c$ on the stellar surface due to the rotation of the magnetic star. To pick up this energy, particles must be able to cross field lines. Radiation of momentum by relativistic electrons is the analogue of the frictional drag on the electron current in a normal plasma, due to scattering by ions: both lead to departure from field freezing. In the present problem, the radiation beyond the LC of photons with high angular momentum enables the electrons to pick up the electrical energy available on field lines of higher potential. Clearly, this requires that the gain of energy exceed the energy carried off by the photons. One can notionally integrate the equation (20) along a trajectory followed by an electron between emission and return to the star, to find the value γ_f with which an electron returns:

$$(\gamma_f - \gamma_i) = -\int \frac{\mathbb{P}}{mc^2}\left[1-\frac{\Omega\tilde{\omega}}{c}\frac{\alpha\tilde{\omega}}{c}\right]\mathrm{d}s \tag{21}$$

(recall that ψ is uniform over the star). Since $\gamma_i \approx 1$, and $\mathbb{P}$ is essentially positive, it is clear that for $\gamma_f > 1$, $(1-(\Omega\tilde{\omega}/c)(\alpha\tilde{\omega}/c))$ must be negative, so that the dissipation must occur primarily beyond the LC (cf. figure 1). This links up with a comment made by Gold (1980) and Holloway (1977). Since it is the star's rotational energy $\frac{1}{2}I\alpha^2$ and

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the associated angular momentum $I\alpha$ that is powering the overall pulsar activity, on balance (energy loss rate) $= \alpha$ (angular momentum loss rate). Photons emitted within the LC, even if they are perfectly beamed in the toroidal direction, have too small a lever arm: there must be compensating photon emission well beyond the LC. The result (20) shows that there is an automatic 'Gold–Holloway' condition not only for the system as a whole but for each trajectory.

Once particles begin to radiate, then a good approximation to the equation of motion balances the Lorentz force against radiation drag, the 'Stokes–Aristotle' approximation:

$$-e(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B}/c)-(\mathbb{P}/c^2)\,\boldsymbol{v}=0. \tag{22}$$

As long as $E<B$, the poloidal velocity $\boldsymbol{v}_{\mathrm{p}}$ remains more nearly parallel to $\boldsymbol{B}_{\mathrm{p}}$, but when $E>B$ it more nearly follows $\boldsymbol{E}$. Strong trans-$\boldsymbol{B}_{\mathrm{p}}$ flow under the essentially quadrupolar $\boldsymbol{E}$-field can be expected close to the equator, where $\boldsymbol{B}_{\mathrm{t}}$ goes to zero by symmetry. Single-particle incoherent radiation has the form

$$\mathbb{P}=\tfrac{2}{3}(e^2/c^3)\,\gamma^4\{\boldsymbol{v}\times(\nabla\times\boldsymbol{v})\}^2. \tag{23}$$

This yields significant trans-$\boldsymbol{B}_{\mathrm{p}}$ flow if $\mathbb{P}\approx ceB$, requiring at points near but beyond the LC that $\gamma\approx\gamma_{\mathrm{d}}$, where

$$\gamma_{\mathrm{d}}^4\approx(\omega_{\mathrm{g}}/\alpha)\,[(c/\alpha)/(e^2/mc^2)], \tag{24}$$

so that with standard values for R and B_{s}, $\gamma_{\mathrm{d}}\approx 1.37\times10^7/P_1^{1/4}$: radiation damping keeps the emitting electrons at energy $\gamma_{\mathrm{d}}\,mc^2\approx 7\times10^{12}/P_1^{1/4}\,\mathrm{eV}$, and the emitted radiation has typical frequency $\nu\approx\gamma_{\mathrm{d}}^3\,\alpha$ by the familiar synchrotron argument – γ-ray photons of energy $\approx 5\times10^7/P_1^{7/4}\,\mathrm{eV}$. The maximum energy available to be picked up by the circulating electrons from the potential difference across the polar cap is $\approx\gamma_{\mathrm{c}}\,mc^2=1.2\times10^{13}/P_1^2\,\mathrm{eV}$, ample for short-period pulsars, but marginal when $P_1\approx 1$ s. The total power is estimated by 2(polar cap area) (potential difference between points of emergence and return) (current density), and turns out to be of the same order as the emission (1) from the oblique vacuum rotator.

Some very approximate analytical and numerical work on this classical model is described in Fitzpatrick & Mestel (1988). To keep the model working the whole system must have zero net charge, so that the electric field well beyond the LC has the quadrupolar $1/r^4$ dependence. Since E must exceed B in the equatorial domain in which the returning electrons have dominantly trans-$\boldsymbol{B}_{\mathrm{p}}$ motions, the toroidal magnetospheric currents are required to yield a dipole moment equal and opposite to that on the star. The essential point of the model is the link-up of the spontaneous loss of energy and angular momentum with the return current requirement. The obvious objection is that (apart from some early Cos-B measurements, subsequently withdrawn), observations show that γ-ray emission is never more than a small fraction of the total energy loss as inferred from spin-down. If the circulating electrons were to radiate coherently, then the maximum γ-values attained could be kept much lower. A cylindrical macroscopic blob containing N electrons feels a radiation drag per particle $\approx Ne^2\gamma^4\alpha^2/c^2$, and this will balance the Lorentz force per particle at much lower values of γ. A rough estimate suggests that if the energy density (electrostatic and kinetic) of the dissipation domain is minimized, γ is limited to $\gamma_{\mathrm{c}}^{\frac{1}{2}}\approx 4\times10^3/P_1$, and the emitted radiation would be at much lower frequencies. However, with the overwhelming evidence that e.g. the Crab pulsar loses much of its spin-down energy in the form of relativistic particles, it is best to look on the classical model as illustrative, with some of its features persisting into the more realistic picture to be discussed below.

3. Models with relativistic acceleration within the light cylinder: pair production

A crucial step in §2 is the requirement that the quasi-MHD flow extends through the LC, which limits the speed with which electrons leave the polar caps typically to $\approx \frac{1}{2}c$. If the emission speeds v^* are somewhat higher, the particles will achieve high γ-values well within the LC. Equations (14) (with the inertial correction negligible) and (17) yield $\rho_e v_p/B_p = \text{const.}$ on a field-streamline. The energy–angular momentum integral (12) can be read together with (3) to show that the GJ approximation – which neglects the non-corotational potential ψ – remains excellent for quite high γ-values, since $|\psi/\phi| \approx \gamma(\alpha/\omega_g)$. Thus by (5), $\rho_e \approx -\alpha B_z/2\pi c$ and so $v_p \propto B/\rho_e \propto B/B_z$, and this increases outwards for field lines that curve away from the axis, as do all the lines in the aligned case and most lines in moderate obliquity cases. Thus if $v^* \lesssim c$, v_p is predicted to reach c on a surface S_1 quite near the star. Beyond S_1 the flow will still follow the field lines closely, but now $v_p \approx c$, and continuity enforces $\rho_e \propto B$ instead of $\rho_e \propto B_z$. Thus the limitation imposed by special relativity shows that the neglect of $\nabla^2\psi$ in the construction of $\nabla \cdot \boldsymbol{E}$ ceases to be valid near and beyond S_1. Equation (5) (with $(\nabla \times \boldsymbol{B})_\phi$ still negligible) is replaced by

$$\nabla^2\psi \approx (2\alpha B_z/c)\,[-1 + (B/B_z)/(B/B_z)_1], \tag{25}$$

so that beyond S_1, ψ is found to increase outwards in order to satisfy the Poisson–Maxwell equation. The integral (12) must now be read in reverse: because the geometry of flow forces B/B_z and so ψ to increase, there is simultaneously an increase in γ which is rapid, again because of the smallness of the ratio α/ω_g. Closer inspection shows that even within S_1 the $\nabla^2\psi$ term is not strictly negligible. The GJ approximation holds in the mean, but superposed on the monotonic flow are stationary spatial oscillations in ψ, γ and v_p of scale $(\alpha/\omega_g)^{\frac{1}{2}}(c/\alpha)$, the plasma wavelength corresponding to the local GJ density. The reason is that the gas will not in general be emitted from the star with the velocity required by the strictly monotonic flow solution (Mestel *et al.* 1985, Appendix B; Shibata 1991). As the gas flows across S_1, the solution with small-scale oscillations in ψ and γ is replaced by one with rapid monotonic growth.

It should be noted that the model is different from those in the literature which appeal to a vacuum gap at the stellar surface. For some years it was thought that in the case of an 'anti-pulsar' (counter-aligned, with the rotation and magnetic axes anti-parallel), the spontaneous emission of ions under the outward pointing electric field would be inhibited, because of a quantum mechanical work function. The 10^{12} V potential difference which builds up at the poles would continually lead to breakdown of the gap by electron–positron pair production (Ruderman & Sutherland 1975). Subsequent computations shed doubt on the previous work-function estimates, and it is now generally assumed that the classical boundary condition $\boldsymbol{E}\cdot\boldsymbol{B} = 0$ is always a good approximation. The present treatment of the aligned case shows that the curvature of the field lines away from the axis necessarily leads to 'over-filling' of the flux-tubes, in the sense that the relativistic limitation on v forces the density up beyond the GJ value, and it is the consequent $E_\parallel$ which causes monotonic acceleration of the electrons to high γ-values. The discussion is consistent with that by Fawley *et al.* (1977), who show that no solutions exist for field lines curving away from the axis if the boundary condition $\boldsymbol{B}\cdot\nabla\psi = 0$ is imposed not only at the star but also far from the star. (Beskin (1990) and Muslimov & Tsygan (1990, 1992) have shown that this conclusion no longer holds when general relativistic corrections are included.) A difference does arise in terminology: in the present treatment this upper boundary condition and so also the phrase 'unfavourably curved' are inappropriate. The position and shape of the surface S_1 is not prescribable locally but must emerge from a global solution. In the limit when S_1 is supposed to be at the pulsar surface, then the model would look as if there were a discontinuity in ψ at the star, but differs from that of Beskin *et al.* (1983), who assume a vacuum gap at the star's surface.

Beyond S_1 the acceleration is so rapid that the one-dimensional approximation for $\nabla^2\psi$ is acceptable, so that (25) and (12) yield

$$\gamma \approx e\psi/mc^2 \approx (e/mc^4)(\alpha^2 B_1/8)(s-s_1)^3, \tag{26}$$

where s is length along a typical field-streamline. When γ reaches $\approx 10^7$ there is again copious emission of gamma rays of energy $h\nu \approx \frac{3}{2}\gamma^3\hbar c/R_c$, where R_c is the radius of curvature of the field-streamline. The crucial new factor, noted first by Goldreich in a conference report and by Sturrock (1971), is that hard γ-rays moving across a very strong magnetic field convert spontaneously into electron–positron ('$e^{\pm}$') pairs. The mean-free-path for pair production by a photon $h\nu > 2mc^2$ is (Erber 1966)

$$l \approx \frac{4}{e^2/\hbar c}\frac{\hbar}{mc}\frac{B_q}{B_\perp}\exp\left(\frac{4}{3\chi}\right), \tag{27}$$

with

$$B_q = m^2c^3/e\hbar = 4.4\times 10^{13}\,G, \quad \chi = (h\nu/2mc^2)(B_\perp/B_q), \tag{28}$$

where $B_\perp$ is the component of $\boldsymbol{B}$ perpendicular to the photon path. The newly created particles are themselves accelerated by the electric field until they radiate γ-rays which again create pairs. This cascade is ultimately self-limiting, since the dense $e^{\pm}$ plasma accompanying the high-γ primary electron current will short out the accelerating $E_\parallel$ field. The basic picture has been applied to different models in many papers (Ruderman & Sutherland 1975; Arons & Scharlemann 1979; Arons 1981; Daugherty & Harding 1982; Jones 1980; among others). A recent approximate discussion, appropriate for the present approach, is by Shibata (1991). A pair creation distance is defined by

$$\begin{aligned} L &= (B_\perp/B)R_c = R_c(2mc^2/h\nu)(B_q\chi)/(B_sR^3/r^3) \\ &= 2.2\times 10^{41}(\alpha r_1/c)^5(R_c/r_1)^2P_1^5/\gamma^3, \end{aligned} \tag{29}$$

where $\hbar\nu = (3/2)\gamma^3\hbar c/R_c$ has been substituted, the magnetic field is again supposed to be dominantly dipolar, with $R_c/r \approx (c/\alpha r)^{\frac{1}{2}}$, and standard values $R = 10^6$ cm and $B_s = 10^{12}$ G are again adopted. As by (27) χ is given as a logarithm and so is slowly varying, a standard value ($\chi = \frac{1}{18}$) may be used. An extreme upper limit for the γ-value attained by the primaries is estimated by inserting L from (29) for $(s-s_1)$ in (26), yielding

$$\gamma \approx 9\times 10^6(r_1/R)^{9/10}P_1^{1/10}. \tag{30}$$

This must again be compared with the maximum possible

$$\gamma \approx (e/mc^2)(\alpha/c)B_sR^2(\alpha R/c) = 2.6\times 10^7 P_1^{-2}$$

available from the potential difference across the polar cap. The ratio is

$\approx 0.4(r_1/R)^{9/10}P_1^{2.1}$, well below unity when $P_1 \ll 1$ even if $r_1/R \gg 1$, but when $P_1 > 1.7$ it will exceed unity even if S_1 is at the star's surface. If the $e^{\pm}$ cascade is essential for the generation of the coherent radio emission from pulsars, then as argued by Sturrock, Ruderman, Sutherland and others, there is a clear physical reason for pulsars' shutting off as their periods lengthen to ≈ 1 s. The multiplicity M (the number of pairs generated per primary electron) is estimated to be from a few hundred for the slower pulsars to $\approx 10^4$ for the Crab pulsar. More precise calculations (Arons 1981; Daugherty & Harding 1982) yield similar results for the multiplicity. The pairs are estimated to have $\gamma \approx 10^2$ and the primary electrons $\gamma \approx 10^6$, markedly below the upper limit (30), so that radiation loss from the primaries is again small after pair production ceases.

Beyond the pair production region $E_\parallel$ is again small, and the $e^{\pm}$ plasma plus the high-γ primary electron gas should behave rather like a normal plasma, with the non-corotational potential $\psi(P)$ again nearly constant on each field line. As long as inertial drifts are small, the flow is again quasi-MHD with

$$\boldsymbol{v} = \kappa \boldsymbol{B} + \tilde{\alpha}\tilde{\omega}\boldsymbol{t}, \quad \tilde{\alpha} = \alpha - c\,\mathrm{d}\psi/\mathrm{d}P, \quad \boldsymbol{E} = -\tilde{\alpha}\tilde{\omega}\boldsymbol{t} \times \boldsymbol{B}/c. \tag{31}$$

The isorotation function $\tilde{\alpha}(P)$ can be $\gtrless \alpha$ (super- or subrotation) according to the trans-$\boldsymbol{B}_\mathrm{p}$ variation of ψ, which in turn depends not only on the detailed physics of the acceleration and pair production domains but also on the position of the surface S_1, and this will emerge strictly only from a global solution in which the different domains are fitted together (cf. §4). As an example, if S_1 is assumed to be an arc of a circle, then the above estimate of γ and so by (26) of ψ predicts subrotation (assumed by Beskin *et al.* (1983)).

As in the pure electron flow within S_1, there is always a small variation of ψ along field streamlines. As long as there is no frictional interaction between the three components, each obeys an energy–angular momentum integral of the form (12), with the density given by continuity, e.g.

$$n_\mathrm{e}\, c(1 - \Omega^2\tilde{\omega}^2/c^2 - 1/\gamma_\mathrm{e}^2)^{\frac{1}{2}} \propto B$$

for the primary electrons. The component $E_\parallel \propto -\boldsymbol{B}\cdot\nabla\psi$ acts to adjust the γ-values, so ensuring that the net charge density $\rho_\mathrm{e} = e(n_+ - n_- - n_\mathrm{e})$ is as required by the Poisson–Maxwell equation $\nabla\cdot\boldsymbol{E} = 4\pi\rho_\mathrm{e}$. Deviations from the quasi-MHD flow (31) become serious when E given by (31) approaches $(B_\mathrm{p}^2 + B_\mathrm{t}^2)^{\frac{1}{2}}$, and simultaneously $\tilde{\alpha}\Omega\tilde{\omega}^2/c^2$ approaches unity and γ for each species begins to grow through the (modified) slingshot term. The force-free approximation for the field structure (cf. §4) begins to break down: instead of the field's controlling the flow of the plasma, the plasma inertia should rather force the field to follow the outflow, analogously to our picture of magnetic stellar winds. The transition will occur for a mean particle γ such that $2Mn_\mathrm{e}\gamma mc^2 \approx B^2/8\pi$, where M is again the multiplicity factor, i.e. for $\gamma \approx (eB/mc\alpha)/8M$. The large factor M now ensures that this value of γ is much below γ_d, given by (24), at which incoherent radiation would rapidly drain the particle energies.

This model with the dense $e^{\pm}$ plasma outflowing with the primary electrons still has to resolve the return current problem. Either there is a compensating positive current elsewhere, or the excess of electrons must leave the outflow, cross field lines and return to the star, again by some dissipative process which carries off angular momentum (the earlier argument against a return current being due to a purely inertial drift in a dissipationless flow carries over). The embarrassing prediction of far

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more γ-radiation than is observed may now be avoided. For example, frictional coupling between primaries and secondaries will act like incoherent radiation in leading to trans-field motion of the circulating electrons, with, however, most of the angular momentum being transferred to the wind rather than emitted as γ-radiation. It has been pointed out by Usov (1987) that because of the actual non-stationary nature of pair creation, the two-stream instability can sometimes couple the high- and low-γ gases. A definitive answer requires study of the outflow under the large-scale electric and magnetic fields – part of the construction of a viable global model.

Energetic electrons returning to the star may also generate further plasma, again either by pair production in the strong $\boldsymbol{B}$-field near the star, or by surface spallation. This could be important, as it is not absolutely clear that the multiplicity factor in the outflow domain will be up to the estimate of 4×10^4–10^5 inferred for injection of leptons into the Crab nebula.

4. Towards the construction of a global model

As long as the kinetic energy density of the gas in any domain is well below $(\boldsymbol{E}^2+\boldsymbol{B}^2)/8\pi$, the $(\boldsymbol{E},\boldsymbol{B})$ fields should jointly satisfy the force-free equation $\rho_e\boldsymbol{E}+(\boldsymbol{j}\times\boldsymbol{B}/c)=0$. Near to the star, $\boldsymbol{B}$ will hardly be distorted from the vacuum dipolar structure. Beyond the acceleration and pair-production domains, equations (31) transform the force-free condition for the outflow domain into

$$\nabla^2P\left[1-\left(\frac{\tilde{\alpha}}{\alpha}\right)^2\left(\frac{\alpha\tilde{\omega}}{c}\right)^2\right]-\frac{2}{\tilde{\omega}}\frac{\partial P}{\partial\tilde{\omega}}+\frac{16\pi^2}{c^2}S\frac{\mathrm{d}S}{\mathrm{d}P}-\left(\frac{\tilde{\alpha}}{\alpha}\right)\frac{\mathrm{d}}{\mathrm{d}P}\left(\frac{\tilde{\alpha}}{\alpha}\right)\left(\frac{\alpha\tilde{\omega}}{c}\right)^2(\nabla P)^2=0, \qquad (32)$$

with the current stream function $S=S(P)$.

Field lines which close within the LC will be part of a dead region, with $S=0$. The simplest example has this region filled with charge-separated gas and corotating with the star: $\tilde{\alpha}/\alpha=1$, and equation (32) reduces to equation (8). The model of Beskin *et al.* (1983) has the outflow region contiguous to the corotating dead zone which extends to the LC, with the return current flowing along the separatrix. The matching of these two zones yields a relation between the current and the non-corotational potential of the outflow zone. In the present picture, with no gap at the stellar surface but with the current and potential fixed just by the position of S_1 and the physics of pair production, a viable model is likely to have a near-vacuum gap, separating dead zone domains in which GJ conditions hold (Holloway 1973; Shibata 1991). Again from equations (31) with $\kappa=0$, in a dead zone not contiguous with the star the isorotation function $\tilde{\alpha}(P)$ depends on the variation of the non-corotational potential ψ between field lines. A gap opening out from the stellar surface will have the equatorial regions super-rotating (Cheng & Ruderman 1991). An approximate wedge-shaped gap well beyond the stellar surface and extending to near the LC has the equatorial regions subrotating (Mestel *et al.* 1985; Fitzpatrick & Mestel 1988), so allowing the dead zone to extend beyond the LC. As noted above, the outflow domain described by equation (32) is likely to be bounded by a surface beyond which the field is dominated by the wind. Significant local deviation from the dissipation-free conditions (31) is also likely. (Further pair production in the gap through photon–photon collisions (Cheng *et al.* 1986) may yield an outward positron current which will contribute to or even resolve the current closure problem.) The task of constructing a self-consistent model is clearly formidable, inevitably involving numerical iteration. Figure 2 is a schematic representation.

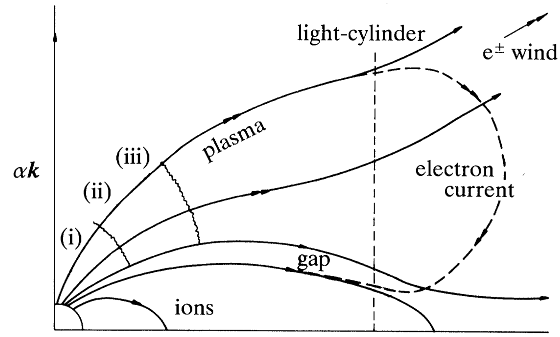

Figure 2. Schematic diagram of the model with pair production. (i) Electrons, (ii) relativistic acceleration, (iii) $e^{\pm}$ production.

The difficulties of an oblique model are still greater, even under the quasi-static assumption $\partial/\partial t = -\alpha\partial/\partial\phi$ (the analogue of $\partial/\partial t$ under axisymmetry). Once an aligned model has been constructed, then the small obliquity case may perhaps be treated as a perturbed model (cf. Mestel & Wang 1982). Some understanding of large obliquity cases can be gained from the cylindrical model in which the z-variation is suppressed (see Burman & Mestel 1979; da Costa & Kahn 1982).

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